ASSOUAD DIMENSION & FRACTAL GEOMETRY

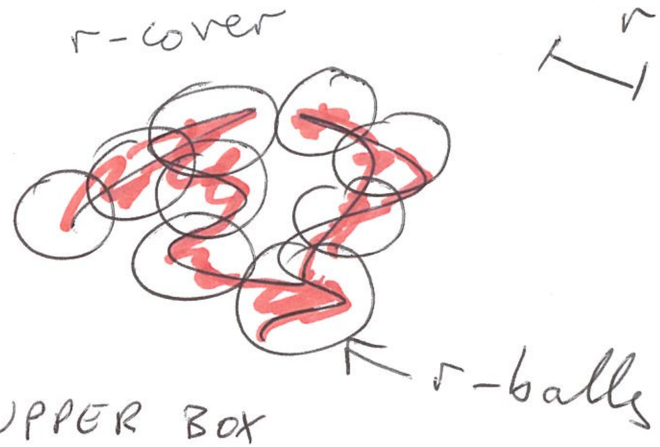
Box dimension: $X \subseteq \mathbb{R}^d$ (bounded)

Q: How "big" is X at small scales?

Fix $r > 0$ (scale)

$N_r(X)$ = "r-covering number" of X.

= min. number of r-balls needed to cover X.



Guess: $N_r(X) \approx r^{-\text{dimension}}$

$$\boxed{\overline{\dim}_B X = \limsup_{r \to 0} \frac{\log N_r(X)}{-\log r}}$$

← UPPER BOX DIMENSION

Assouad dimension:

key idea: "local" box dimension

Fix $0 < r < R \; (< 1)$

covering scale (r)

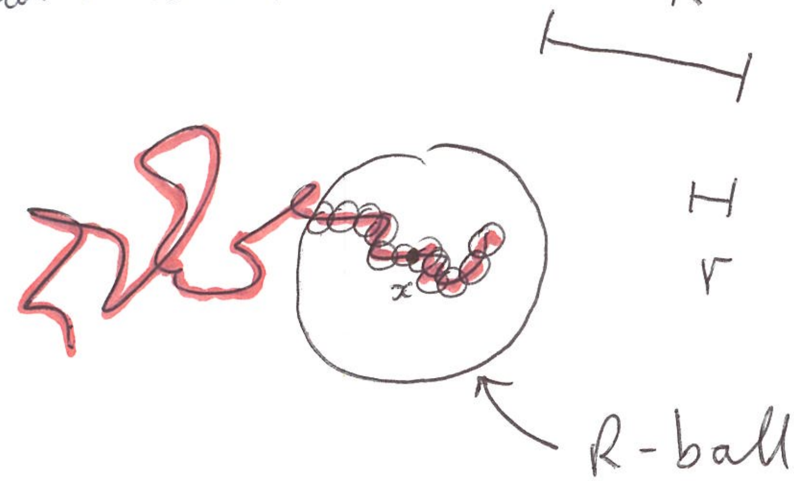
localisation scale (R)

Fix $x \in X$, cover $B(x,R) \cap X$.

$$\inf \left\{ s > 0 : \exists C > 0 : \forall x \; \forall 0 < r < R : \quad N_r\big(B(x,R) \cap X \big) \leq C \left(\frac{R}{r} \right)^s \right\}$$

$= \dim_A X$. $\boxed{\text{Assouad dimension}}$

local r-cover:



Example: $X = \{ 1/n : n \in \mathbb{N} \} \cup \{0\} \subseteq \mathbb{R}$

$$\overline{\dim}_B X = 1/2$$

Gap size: $\frac{1}{n} - \frac{1}{n+1} \approx \frac{1}{n^2} \approx r$

$$N_r(X) \approx 1 + r^{-1/2} \approx r^{-1/2}$$

$\dim_H X = 0$ (all countable sets have Hausdorff dimension 0).

$\overline{\dim}_B X = 1/2$

$\dim_A X = 1$

Proof: $\dim_A X \leq 1$ is clear.

For lower bound: fix $s \in (0,1)$

want to show: $\forall C > 0 \ \exists x \in X \ \exists 0 < r < R < 1$:

$$N_r(B(x,R) \cap X) > C\left(\frac{R}{r}\right)^s$$

Get the logic right

Fractal Geometry

Let $C > 0$. Choose $x = 0$. Choose $R > 0$ small.

Choose $r = R^2$.

All gaps below R are $\leq R^2$.

Analysis

Therefore $N_r(B(0,R) \cap X) \approx \dfrac{R}{r} > C\left(\dfrac{R}{r}\right)^s$

for R small enough.

same as covering whole interval $[0,R]$

Therefore $\dim_A X \geq 1$.

Basic properties:

① Monotone: $X \subseteq Y \Rightarrow \dim_A X \le \dim_A Y$

② Open sets: $X \subseteq \mathbb{R}^d$ open & non-empty $\Rightarrow \dim_A X = d$.

③ bi-Lipschitz property: $f: \mathbb{R}^d \to \mathbb{R}^d$ $\underbrace{\text{bi-Lipschitz}}$ $\Rightarrow \dim_A f(X) = \dim_A X \quad \forall X \subseteq \mathbb{R}^d$.

$$\frac{1}{C}|x-y| \le |f(x)-f(y)| \le C|x-y|$$

④ Lipschitz property: It is NOT TRUE that!

f Lipschitz $\Rightarrow \dim_A f(X) \le \dim_A X$.

⑤ Product sets: $X \subseteq \mathbb{R}$, $Y \subseteq \mathbb{R}$

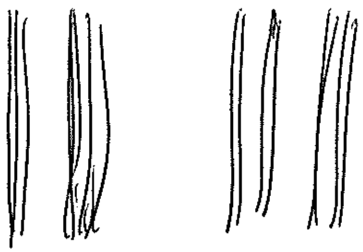
Then $X \times Y = \{(x,y) : x \in X, y \in Y\}$

$\subseteq \mathbb{R}^2$

e.g. X = Cantor set

$Y = [0,1]$

$X \times Y$



Guess:

$\dim X \times Y = \dim X + \dim Y$?

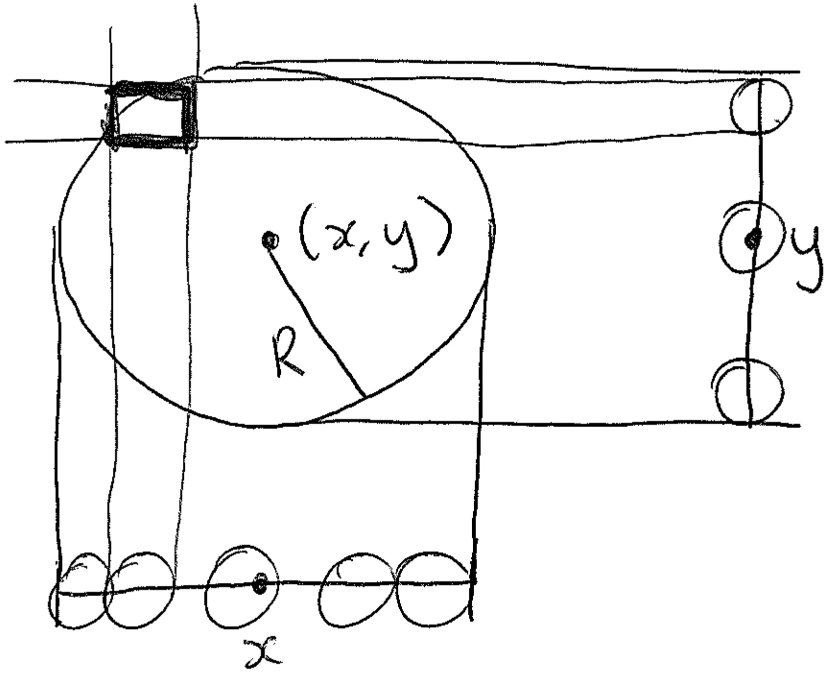
<u>Fact</u>: $\dim_A X \times Y \leq \dim_A X + \dim_A Y$.

$\dim_A X \times Y \geq \dim_A X + \dim_L Y$

"lower dimension"

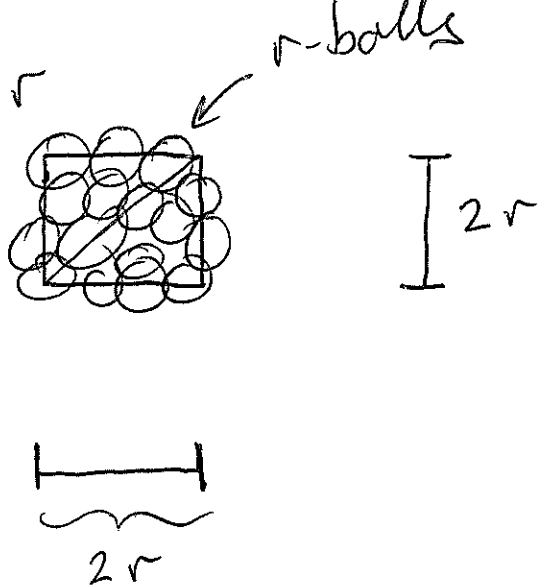
Let $s > \dim_A X$, $t > \dim_A Y$

Let $(x,y) \in X \times Y$, $0 < r < R$ (arbitrary).



r-cover of $Y \cap B(y,R)$ by $\leq C\left(\frac{R}{r}\right)^t$ r-balls.

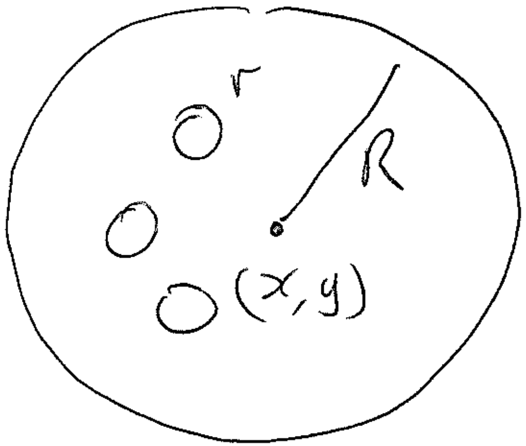
r-cover of $X \cap B(x,R)$ by $\leq C\left(\frac{R}{r}\right)^s$ r-balls

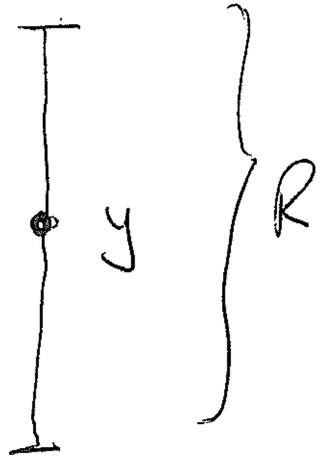


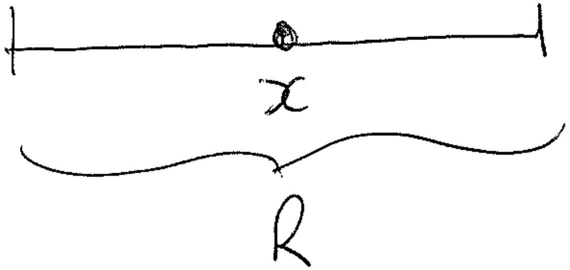
Build cover of $X \times Y$ with

$$\leq C\left(\frac{R}{r}\right)^s \times C\left(\frac{R}{r}\right)^t \times 20 = 20C^2\left(\frac{R}{r}\right)^{s+t},$$ as required.

why not $\dim_A X \times Y \geq \dim_A X + \dim_A Y$?







Cannot guarantee

$N_r(B(x,R)\cap X)$ and

$N_r(B(y,R)\cap Y)$ both big

for same $0<r<R$.

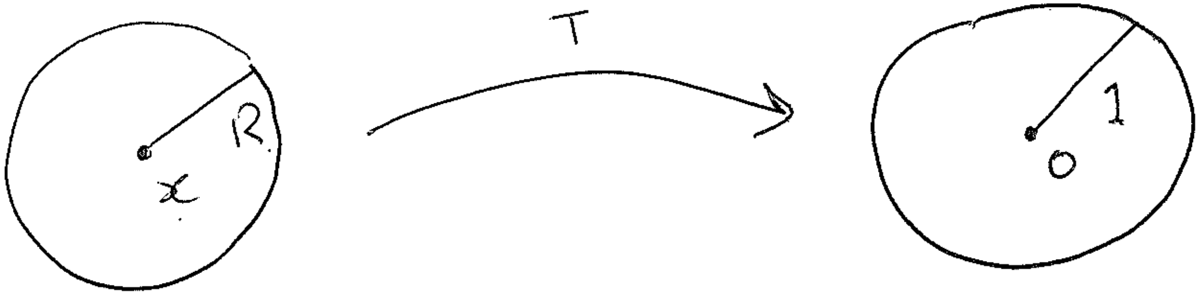
Tangents:

think of tangents of a differentiable curve

To define a tangent, we need to "localise" and use a suitable "notion of convergence."

$X \subseteq \mathbb{R}^d$ compact, non-empty

To localise: choose $x \in X$, $R > 0$. Then



Consider $T(X) \cap B(0,1)$ (a compact set).

Apply T: $T(z) = \frac{1}{R}(z - x)$

For $X, Y \subseteq \mathbb{R}^d$ compact, non-empty

$$d_{\mathcal{H}}(X, Y) = \inf \left\{ \delta > 0 : \ X \subseteq Y_\delta, \ Y \subseteq X_\delta \right\}$$

Hausdorff metric

$$X_\delta = \text{"}\delta\text{-neighbourhood" of } X$$

$$= \left\{ z \in \mathbb{R}^d : \ \exists x \in X : \ |x - z| < \delta \right\}$$

Then $\left(K(\mathbb{R}^d), d_{\mathcal{H}} \right)$ is a complete metric space.

non-empty compact subsets of $\mathbb{R}^d$

Definition: E is a <u>weak tangent</u> to X if there exits a sequence x_n, R_n such that

$$T_n(X) \cap B(0,1) \to E \text{ in } d_H.$$

$$\frac{1}{R_n}(X - x_n) \nearrow T_n(X)$$

Theorem (Mackay–Tyson 2011)

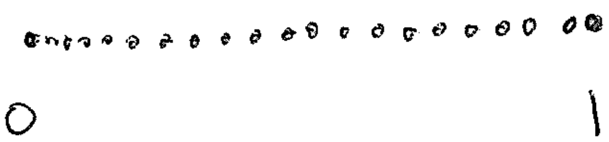
If E is a weak tangent to X, then

$$\dim_A X \geq \dim_A E.$$

(useful because E may be very simple or regular) e.g. $\quad X = \{ 1/n : n \in \mathbb{N} \} \cup \{0\}$

Consider $\quad mX \cap [0,1] = \{ m/n : n \geq m \} \cup \{0\}$

$mX \cap [0,1]$:



$\xrightarrow{d_H} [0,1]$.

Therefore $\quad \dim_A X \geq \dim_A [0,1] = 1$.

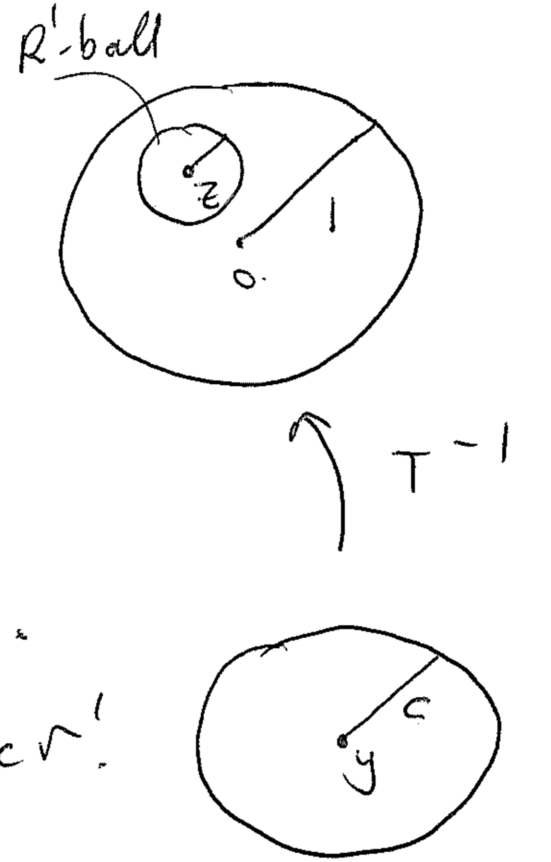
($[0,1]$ is "simple" and easy to study)

Proof: $\underbrace{\text{Let } s > \dim_A X}$. Want to show $\dim_A E \le s$.

$$\Rightarrow \exists\, C>0 \mid \forall x \in X \ \forall\, 0<r<R \quad N_r(B(x,R)\cap X) \le C\left(\frac{R}{r}\right)^s$$

Fix $z \in E$, $0 < r' \le R' < 1$

Build cover of $B(z,R') \cap E$



Find $\frac{r}{2}$-approximation of E by

$$T(X) \cap B(0,1) = \frac{1}{c}(X-y) \cap B(0,1).$$

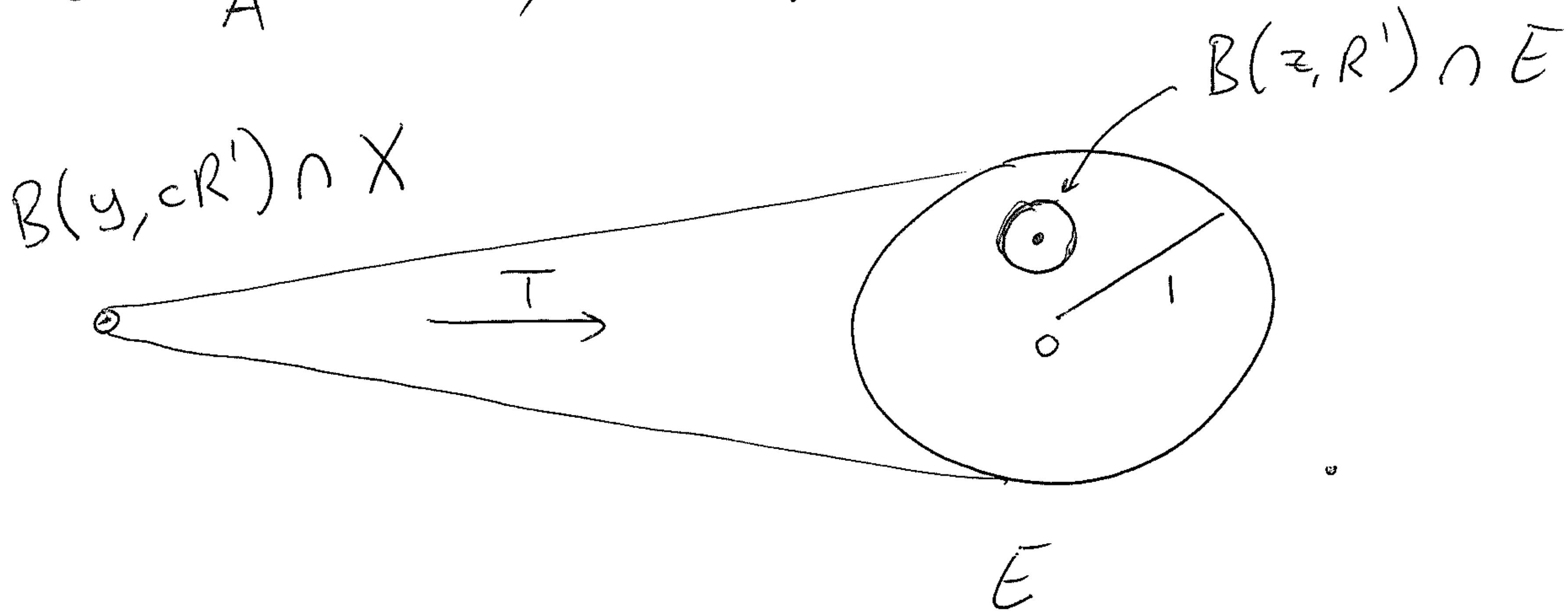
Then, cover $B(y, cR') \cap X$ at scale cr'

by $\le C\left(\frac{\cancel{c}R'}{\cancel{c}r'}\right)^s$ many sets.

Therefore, I have r'-cover of $B(z, R') \cap E$

by $\leq C\left(\frac{2R'}{r'/2}\right)^s$ r'-balls

$\Rightarrow \dim_A E \leq s$, as required.



Theorem (Käenmäki – Ojala – Rossi: 2018 IMRN)
($X \subseteq \mathbb{R}^d$, compact)

$$\dim_A X = \sup \left\{ \dim_H E : E \text{ is a weak tangent to } X \right\}$$

Note: • $\dim_A X \geq \dim_A E \geq \dim_H E$ for all weak tangents E. (By Mackay-Tyson)

(• "sup" is in fact "max"

• Can even find weak tangent with positive $\dim_A X$-dimensional Hausdorff measure.)

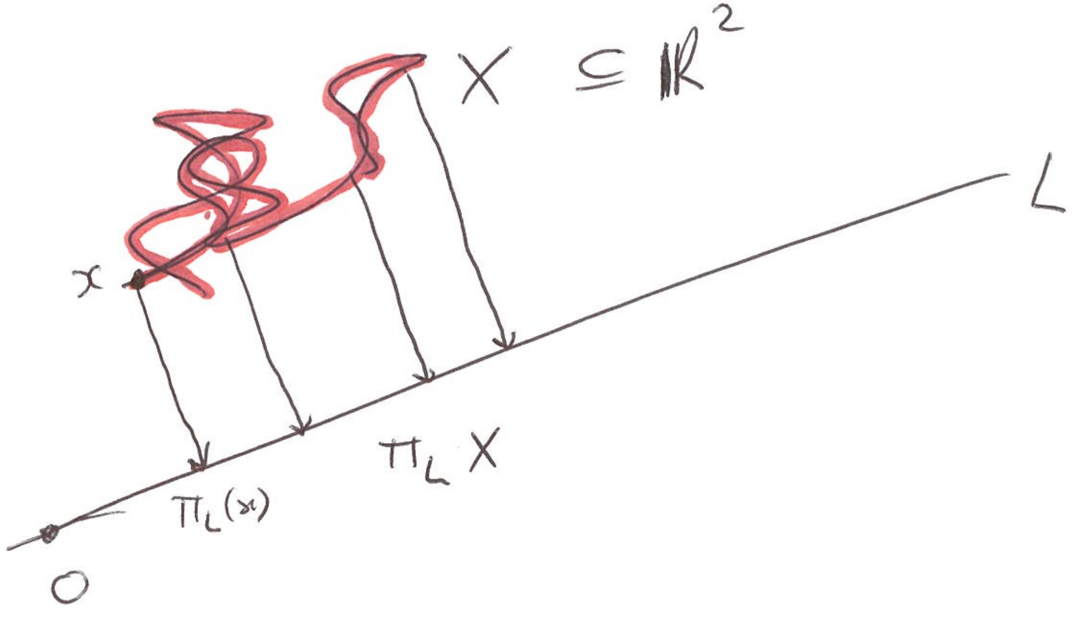
Projections and dimension

Marstrand 1954

Mattila 1970s

Kaufman. 1968

$\cancel{X \subseteq \mathbb{R}^d}$ $X \subseteq \mathbb{R}^2$, consider projections of $\mathbb{R}^2$ onto lines.



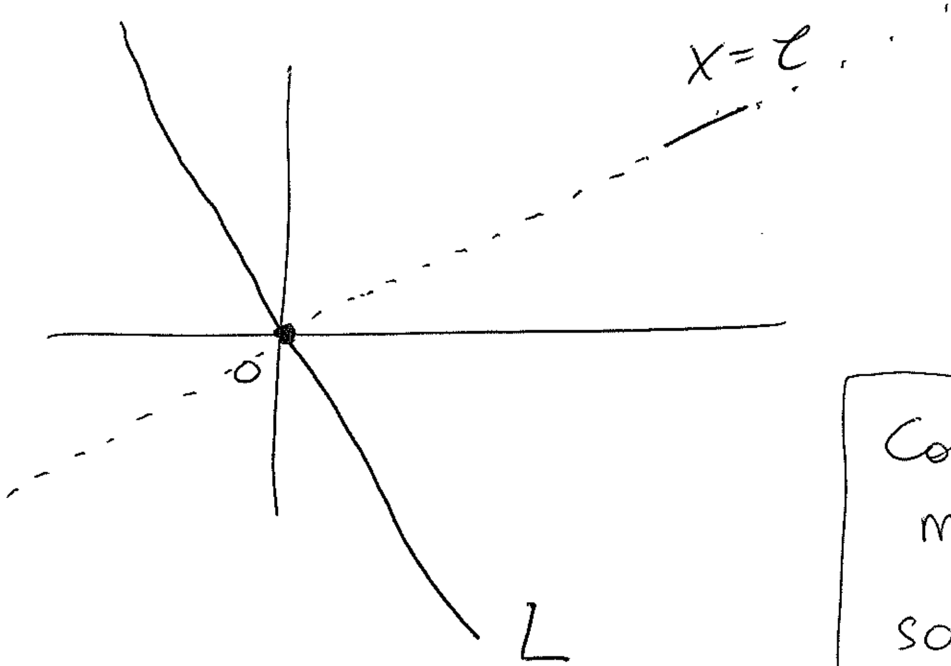
π_L for orthogonal projection onto L.

$\pi_L X \subseteq L \equiv \mathbb{R}$

Question: how to relate $\dim \pi_L X$ and $\dim X$?

e.g. $X = \ell$ a line segment

$\pi_L(X) = \{0\}$ for L orthogonal to ℓ.



$\dim X = 1$

$\dim \pi_L X = 0$.

Conclusion: we cannot say much for <u>all</u> X and <u>all</u> L, so we try to describe "generic" L.

Marstrand Projection Theorem (1954)

$X \subseteq \mathbb{R}^2$ compact. (or Borel)

$$\dim_H \pi_L X = \min\{1, \dim_H X\}$$

for ~~all~~ almost all lines L.

<u>Notes</u>: • $\dim_H \pi_L X \leq \min\{1, \dim_H X\}$ for all L.

• "almost all" refers to 1D Lebesgue measure

<u>Proof</u> Potential theoretic method
+ transversality.

Theorem (Fraser-Orponen PLMS 2017)

$X \subseteq \mathbb{R}^2$ compact.

$\dim_A \pi_L X \geqslant \min\{1, \dim_A X\}$ for almost all L.

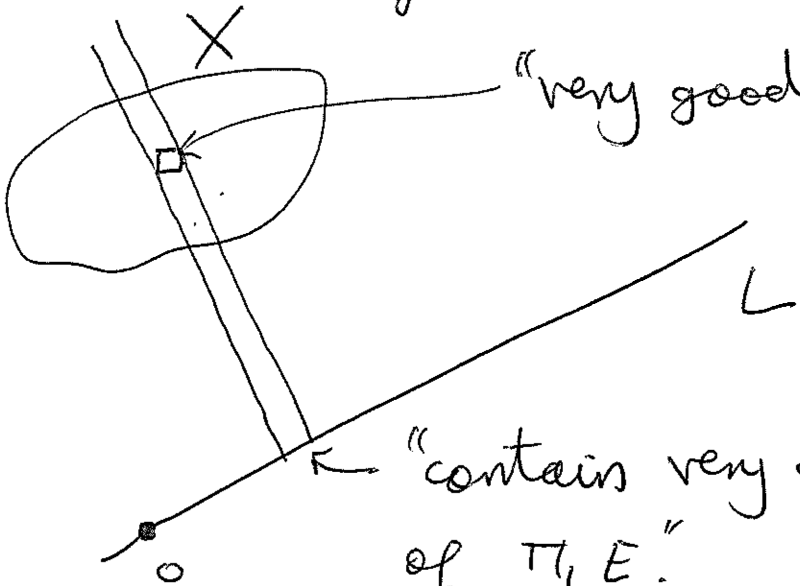
① "$\leqslant$" does not hold! (see Fraser-Käenmäki PAMS 2020)

② "almost all" can be upgraded to "all but an exceptional set of Hausdorff dimension zero" (Orponen PLMS 2021)

③ See Fraser (Israel 2018) for higher dimensional case.

Proof: $X \subseteq \mathbb{R}^2$. Use Käenmäki–Ojala–Rossi to find weak tangent E to X with $\dim_H E = \dim_A X$.

claim: $\pi_L E$ is contained in a weak tangent of $\pi_L X$ for all L.



"very good approximation of E".

← "contain very good approximations of $\pi_L E$."

generate sequence of increasingly good approximations

Use fact that $(K(B(0,1)), d_{\mathcal{H}})$ is compact to extract convergent subsequence giving a weak tangent F to $\pi_L X$ which contains $\pi_L E$.

Then

$$\dim_A \pi_L X \geq \dim_H F \qquad \text{(Mackay-Tyson)}$$

$$\geq \dim_H \pi_L E \qquad \text{(Monotonicity)}$$

for almost all L.

$$\geq \min\{1, \dim_H E\} \qquad \text{(Marstrand)}.$$

$$= \min\{1, \dim_A X\}.$$

Assouad dimension & Fractal Geometry

Exercises

① Let $X_p = \{1/n^p : n \in \mathbb{N}\} \cup \{0\}$ where $p > 0$ is fixed.
Prove that $\dim_B X_p = \frac{1}{1+p}$

and $\dim_A X_p = 1$.

② Let $X = \{2^{-n} : n \in \mathbb{N}\} \cup \{0\}$.
Prove that $\dim_A X = 0$.

③ Let $X = \{2^{-\sqrt{n}} : n \in \mathbb{N}\} \cup \{0\}$.
calculate $\dim_A X$?

④ Constuct examples $E, F \subseteq [0,1]$ such that

$$\dim_A E \times F < \dim_A E + \dim_A F.$$

⑤ Construct $X \subseteq \mathbb{R}^2$ compact and $f : \mathbb{R}^2 \to \mathbb{R}^2$ Lipschitz such that

$$\dim_A f(X) > \dim_A X.$$

⑥ Show that $f : (K(\mathbb{R}^d), d_{\mathcal{H}}) \to \mathbb{R}$ defined by $f(X) = \dim_A X$ is not a continuous function.

⑦ Let f be as in the previous question. Show f is Borel measurable.

⑧ Show that "tangents are not enough". That is, construct $X \subseteq [0,1]$ compact such that $\dim_A X > 0$ but $\dim_A E = 0$ for all E obtained as limits of

$$\frac{1}{R_n}(X - x) \cap B(0,1) \text{ for } R_n \text{ varying and } x \text{ fixed.}$$

(9) Prove that $\dim_A X \le d$ for all $X \subseteq \mathbb{R}^d$.

(10) Prove that $\dim_A X = \dim_A \overline{X}$ for all $X \subseteq \mathbb{R}^d$ where $\overline{X}$ is the closure of X.

(11) Let $f, g : [0,1] \to \mathbb{R}$ be continuous functions. Prove that

$$\overline{\dim}_B G_{f+g} \le \max\{\overline{\dim}_B G_f, \overline{\dim}_B G_g\}$$

where $(f+g)(x) = f(x) + g(x)$ and $G_f = \{(x, f(x)) : x \in [0,1]\}$ is the "graph" of f.

(12) Construct examples of continuous functions $f, g : [0,1] \to \mathbb{R}$ such that

$$\dim_A G_{f+g} > \max\{\dim_A G_f, \dim_A G_g\}.$$

(13) Construct examples of continuous functions $f, g : [0,1] \to \mathbb{R}$ such that

$$\dim_H G_f = \dim_H G_g = 1$$

but $\dim_H G_{f+g} = 2$.

[Hint: use Weierstrass approximation theorem and Baire category theorem]

Why does Baire Category not work for Assouad dimension?